F Optimization

◆ **Optimization problems:**

➤ Minimization of cost for given throughput

➤ Maximization of throughput with cost constraint

➤ Minimization of the mean response time with cost constraint

◆ **Decision variables:**

➤ System parameters that can be varied to obtain optimal results

➤ For computer systems service rates are often decision variables

◆ Cost functions:

➤ Linear cost function (c_i = cost factor for service rate μ_i):

$$C(\mu) = \sum_{i=1}^{N} c_i \mu_i = \mathrm{COST}$$

➤ Non linear cost function:

$$C(\mu) = \sum_{i=1}^{N} c_i \mu_i^{\alpha_i} = \mathrm{COST}\,, \quad \alpha_i > 1$$

➤ Cost function including the costs for main memory *C(K)*:

$$C(\mu, K) = C(K) + \sum_{i=1}^{N} c_i \mu_i^{\alpha_i} = \mathrm{COST}$$

Optimization based on Summation method:

◆ Fundamental equation of the summation method:

$$f_i(\lambda_i) = \overline{K}_i = \begin{cases} \dfrac{\rho_i}{1 - \dfrac{K-1}{K}\rho_i}, & \text{Type-1,2,4 } (m_i = 1), \\ \dfrac{\lambda_i}{\mu_i}, & \text{Type-3.} \end{cases}$$

➤ Simplified for optimization (without correction factor):

$$\overline{K}_i(\lambda, \mu_i) = f_i(\lambda, \mu_i) = \begin{cases} \dfrac{\lambda e_i}{\mu_i - \lambda e_i}, & \text{Type-}\mathbf{1, 2, 4}\text{, and } m_i = 1, \\ \dfrac{\lambda e_i}{\mu_i}, & \text{Type-3 IS.} \end{cases}$$

➤ System equation:

$$\sum_{i=1}^{N} f_i(\lambda, \mu_i) = \sum_{\neq \text{IS}} \frac{\lambda e_i}{\mu_i - \lambda e_i} + \sum_{\text{IS}} \frac{\lambda e_i}{\mu_i} = K\,.$$

➤ Results are approximate (system equation $\rightarrow$ approximation),

◆ **Maximization of the Throughput λ with linear cost function**:

➤ Lagrange function (y_1, y_2 : Lagrange Multiplier):

$$L(\lambda, \mu_1, \ldots, \mu_N, y_1, y_2) = \lambda + y_1\left(\sum_{i=1}^{N} c_i\mu_i - \mathrm{COST}\right) + y_2\left(\sum_{\neq \mathrm{IS}} \frac{\lambda e_i}{\mu_i - \lambda e_i} + \sum_{\mathrm{IS}} \frac{\lambda e_i}{\mu_i} - K\right)$$

➤ A necessary condition for optimal service rates μ_i and maximum throughput λ is obtained by differentiating with respect to λ, μ_i, y_1, y_2:

$$\frac{\partial L}{\partial \lambda} = 0\,, \qquad \frac{\partial L}{\partial y_1} = 0\,,$$
$$\frac{\partial L}{\partial \mu_i} = 0\,, \quad i = 1, \ldots, N\,, \qquad \frac{\partial L}{\partial y_2} = 0\,.$$

➤ The optimal values are obtained by solving the preceding system of equations:

$$\lambda^* = \frac{\mathrm{COST}\cdot K}{\left(\sum_{i=1}^{N}\sqrt{c_i e_i}\right)^2 + K\sum_{\neq \mathrm{IS}} c_i e_i}\,,$$

$$\mu_i^* = \begin{cases} \lambda^*\cdot e_i\left(\dfrac{\sum_{j=1}^{N}\sqrt{e_j c_j}}{K\sqrt{e_i c_i}} + 1\right), & \text{type } i \neq \mathrm{IS},\\[2ex] \lambda^*\cdot e_i\left(\dfrac{\sum_{j=1}^{N}\sqrt{e_j c_j}}{K\sqrt{e_i c_i}}\right), & \text{type } i = \mathrm{IS}. \end{cases}$$

◆ Minimization of cost subject to a minimum throughput requirement:

➤ Optimal values of the service rates:

$$\mu_i^* = \begin{cases} \lambda e_i \left(\dfrac{\sum\limits_{j=1}^{N} \sqrt{e_j c_j}}{K \sqrt{e_i}} + 1 \right), & \text{type } i \neq \text{IS}, \\ \lambda e_i \left(\dfrac{\sum\limits_{j=1}^{N} \sqrt{e_j c_j}}{K \sqrt{e_i}} \right), & \text{type } i = \text{IS}. \end{cases}$$

➤ Minimal cost:

$$C^*(\mu) = \sum_{i=1}^{N} \mu_i^* c_i \,.$$

◆ Minimization of the cost using a non linear cost function:

➤ Initialization:

$$\mu_i = 1, \text{ for } i = 1, \ldots, N$$

➤ Iteration:

$$y = \lambda \cdot \left(\frac{1}{K} \sum_{i=1}^{N} \sqrt{\alpha_i c_i e_i \mu_i^{\alpha_i - 1}} \right)^2 ,$$

$$\mu_i = \begin{cases} \sqrt{\dfrac{\lambda y e_i}{\alpha_i c_i \mu_i^{\alpha_i - 1}}} + \lambda e_i \,, & \text{type } i \neq \text{IS}\,, \\ \left(\dfrac{\lambda y e_i}{\alpha_i c_i} \right)^{\frac{1}{\alpha_i - 1}} & \text{type } i = \text{IS}\,. \end{cases}$$

◆ **Example**: Closed product form queueing network: **Maximization** of the **throughput** using a linear cost function

➤ K = 20

➤ N = 5

– 3 M/M/1-FCFS

– 2 M/G/∞–IS

➤ Visit ratios:

$$e_1 = 1\,,\quad e_2 = 0.2\,,\quad e_3 = e_4 = 0.5\,,\quad e_5 = 0.3\,.$$

➤ Cost factors:

$$c_1 = 10\,,\quad c_2 = c_3 = 5\,,\quad c_4 = 2\,,\quad c_5 = 1\,.$$

➤ Cost constraint:

$$\mathrm{COST} = 100$$

➤ Results: $\lambda^* = 6.189$; $\lambda_{MVA} = 6.569$

$$\mu_1^* = 6.189\,,\quad \mu_2^* = 1.689\,,\quad \mu_3^* = 3.812\,,\quad \mu_4^* = 1.096\,,\quad \mu_5^* = 1.236\,,$$

➤ 3 additional examples:

	Example 1		Example 2		Example 3	
	BFS Method	Complex Method	BFS Method	Complex Method	BFS Method	Complex Method
μ_1^*	6.90	6.81	8.07	7.97	3.04	3.00
μ_2^*	1.69	1.65	1.30	1.41	0.66	0.69
μ_3^*	3.81	3.99	2.83	2.95	1.27	0.80
μ_4^*	1.13	1.01	0.90	0.93	0.59	0.79
μ_5^*	1.24	1.50	1.10	1.13	0.67	0.84
λ_{BFS}^*	6.19	6.56	6.66	7.56	2.63	2.95
λ_{MVA}	6.57	6.55	7.54	7.57	3.00	2.97
COST	100.00	99.81	100.00	99.97	100.00	100.00

➤ Non linear cost function:

α	1	1.25	1.5	2.0	2.5	3
μ_1^*	6.912	4.911	3.884	2.860	2.361	2.067
μ_2^*	1.689	1.242	1.017	0.806	0.716	0.674
μ_3^*	3.182	2.727	2.173	1.625	1.363	1.214
μ_4^*	1.096	0.921	0.821	0.735	0.706	0.704
μ_5^*	1.236	0.998	0.883	0.781	0.774	0.732
λ_{BFS}^*	6.189	4.438	3.532	2.625	2.180	1.918
λ_{MVA}	6.596	4.711	3.752	2.790	2.319	2.0 39

Optimization based on the convolution algorithm:

◆ Maximization of the throughput:

➤ Convolution algorithm and BCMP theorem:

$$\lambda(\mu, K) = \frac{G(\mu, K-1)}{G(\mu, K)}$$

$$G(\mu, K) = \sum_{\sum_{i=1}^{N} k_i = K} \prod_{i=1}^{N} F_i(k_i)$$

$$F_i(k_i) = \left(\frac{e_i}{\mu_i}\right)^{k_i} \cdot \frac{1}{\beta_i(k_i)} \qquad \beta_i(k_i) = \begin{cases} k_i!\,, & k_i \le m_i\,, \\ m_i!\,m_1^{k_i - m_i}\,, & k_i \ge m_i\,, \\ 1\,, & m_i = 1\,. \end{cases}$$

➤ Cost function:

$$C(\mu, K) = C(K) + \sum_{i=1}^{N} c_i \mu_i^{\alpha_i} = \mathrm{COST}$$

➤ Lagrange function:

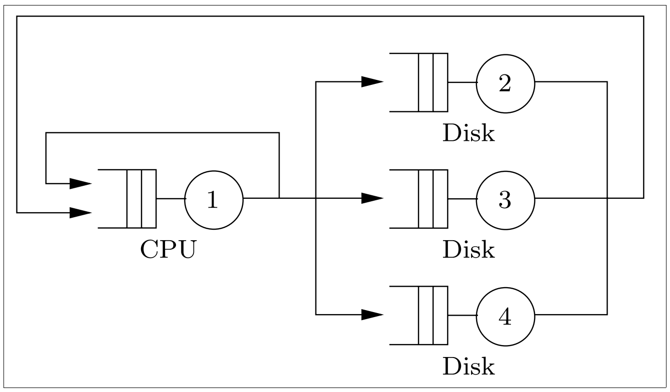
$$L(\mu, y, K) = \lambda(\mu, K) + y\left(C(K) - \mathrm{COST} + \sum_{i=1}^{N} c_i \mu_i^{\alpha_i}\right)$$

➤ Differentiation:

$$\frac{\partial L}{\partial \mu_i} = 0\,, \qquad i = 1, \ldots, N\,,$$
$$\frac{\partial L}{\partial y} = 0\,.$$

➤ We can use the computer algebra program **MAPLE** to obtain the derivatives and solve the non-linear system of equations.

➤ **Example**:



➤ Parameter: COST = 500; $C(K) = C_m K$; $C_m = 5$

➤ Cost factors and exponents:

	c_i	α_i
CPU	131.9	0.55
Disk_1	11.5	1.00
Disk_2	54.2	0.67
Disk_2	54.2	0.67

➤ Results:

K	λ^*	μ_1^*	μ_2^*	μ_3^*	μ_4^*
1	0.071	3.090	4.240	2.054	1.376
2	0.091	2.819	3.373	1.525	1.032
3	0.089	2.356	2.684	1.238	0.752
4	0.076	1.967	1.979	0.847	0.571
5	0.059	1.557	1.084	0.598	0.431